

BOUNDARY INTERPOLATION BY FINITE BLASCHKE PRODUCTS

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ABSTRACT. Given n distinct points t_1, \dots, t_n on the unit circle \mathbb{T} and equally many target values $w_1, \dots, w_n \in \mathbb{T}$, we describe all Blaschke products f of degree at most $n - 1$ such that $f(t_i) = w_i$ for $i = 1, \dots, n$. We also describe the cases where degree $n - 1$ is the minimal possible.

1. INTRODUCTION

Let \mathcal{S} denote the Schur class of all analytic functions mapping the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ into the closed unit disk $\overline{\mathbb{D}}$. We denote by \mathcal{RS}_k the set of all rational \mathcal{S} -class functions of degree at most k . The functions $f \in \mathcal{RS}_k$ that are unimodular on the unit circle \mathbb{T} are necessarily of the form

$$f(z) = c \cdot \prod_{i=1}^k \frac{z - a_i}{1 - \overline{z} \overline{a_i}}, \quad |c| = 1, \quad |a_i| < 1$$

and are called *finite Blaschke products*. They can be characterized as Schur-class functions that extend to a mapping from $\overline{\mathbb{D}}$ onto itself and then the degree $k = \deg f$ can be interpreted geometrically as the winding number of the image $f(\mathbb{T})$ of the unit circle \mathbb{T} about the origin (i.e., the map $f : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$ is k -to-1). We will write \mathcal{B}_k for the set of all Blaschke products of degree at most k , and we will use notation $\mathcal{B}_k^\circ := \mathcal{B}_k \setminus \mathcal{B}_{k-1}$ for the set of Blaschke products of degree k .

Given points $z_1, \dots, z_n \in \mathbb{D}$ and target values $w_1, \dots, w_n \in \overline{\mathbb{D}}$, the classical *Nevanlinna-Pick problem* consists of finding a function $f \in \mathcal{S}$ such that

$$f(z_i) = w_i \quad \text{for } i = 1, \dots, n \quad (z_i \in \mathbb{D}, w_i \in \overline{\mathbb{D}}). \quad (1.1)$$

The problem has a solution if and only if the matrix $P = \left[\frac{1 - w_i \overline{w_j}}{1 - z_i \overline{z_j}} \right]_{i,j=1}^n$ is positive semidefinite ($P \geq 0$) [17, 16]. If $\det P = 0$, the problem has a unique solution which is a finite Blaschke product of degree $k = \text{rank } P$. If P is positive definite ($P > 0$), the problem is *indeterminate* (has infinitely many solutions), and its solution set admits a linear-fractional parametrization with free Schur-class parameter. When the parameter runs through the class \mathcal{B}_κ° , the parametrization formula produces all Blaschke-product solutions to the problem (1.1) of degree $n + \kappa$ for each fixed $\kappa \geq 0$.

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If the problem (1.1) is indeterminate, it has no solutions in \mathcal{B}_{n-1} . However, it still has solutions in \mathcal{RS}_{n-1} (*low-degree solutions*). In case not all target values w_i 's are the same, the problem has infinitely many low-degree solutions which can be parametrized by polynomials σ with $\deg \sigma < n$ and with all the roots outside \mathbb{D} ; see [7], [10], [11]. More precisely, for every such σ , there exists a unique (up to a common unimodular constant factor) pair of polynomials $a(z)$ and $b(z)$, each of degree at most $n-1$ and such that

- (1) $|a(z)|^2 - |b(z)|^2 = |\sigma(z)|^2$ for $|z| = 1$ and
- (2) the function $f = b/a$ (which belongs to \mathcal{RS}_{n-1} by part (1)) satisfies (1.1).

The question of finding a rational solution of the minimal possible degree k_{\min} (and even finding the value of k_{\min}) is still open.

The boundary version of the Nevanlinna-Pick problem interpolates preassigned values w_1, \dots, w_n (interpreted as nontangential boundary limits in the non-rational case) at finitely many points t_1, \dots, t_n on the unit circle \mathbb{T} . Obvious necessary conditions $|w_i| \leq 1$ ($1 \leq i \leq n$) turn out to be sufficient, and a solvable problem is always indeterminate. If at least one of the preassigned boundary values w_i is not unimodular, the problem cannot be solved by a finite Blaschke product. However, as in the classical “interior” case, the problem admits infinitely many low-degree rational solutions $f \in \mathcal{RS}_{n-1}$ (unless all target values w_i 's are equal to each other); see e.g., [4]. As for now, the description of all low-degree solutions is not known.

In this paper we will focus on the “boundary-to-boundary” Nevanlinna-Pick problem where all preassigned boundary values are unimodular. This problem can be solved by a finite Blaschke product.

Theorem 1.1. *Given any points $t_1, \dots, t_n \in \mathbb{T}$ and $w_1, \dots, w_n \in \mathbb{T}$, there exists a finite Blaschke product $f \in \mathcal{B}_{n-1}$ such that*

$$f(t_i) = w_i \quad \text{for } i = 1, \dots, n \quad (t_i, w_i \in \mathbb{T}). \quad (1.2)$$

Any rational function $f \in \mathcal{RS}_{n-1}$ satisfying conditions (1.2) is necessarily a Blaschke product (i.e., $f \in \mathcal{B}_{n-1}$).

The existence of a finite Blaschke product satisfying conditions (1.2) was first confirmed in [8] with no estimates for the minimal possible degree k_{\min} of f . The estimate $k_{\min} \leq n^2 - n$ was obtained in [22]. Ruscheweyh and Jones [19] improved this estimate to $k_{\min} \leq n-1$, which is sharp for some problems. Several different approaches to constructing interpolants in \mathcal{B}_{n-1} have been presented in [14, 21, 15]. The paper [21] also contains interesting results concerning minimal degree solutions. The fact that all low-degree solutions to the problem (1.2) are necessarily Blaschke products was established in [4].

If $w_i = w \in \mathbb{T}$ for $i = 1, \dots, n$, then the constant function $f(z) \equiv w$ is the only \mathcal{B}_{n-1} -solution to the problem; it is not hard to show that such a problem has no other *rational* solutions of degree less than n . Otherwise (i.e., when at least two target values are distinct), the set of all \mathcal{B}_{n-1} -solutions (or, which is the same, the set of all \mathcal{RS}_{n-1} -solutions) to the problem (1.2) is infinite. The parametrization of this set is presented in Theorem 3.5, the main result of the

paper. In Theorem 3.6 we give a slightly different parametrization formula which is then used in Section 4 to characterize the problems (1.2) having no Blaschke product solutions of degree less than $n - 1$.

2. THE MODIFIED INTERPOLATION PROBLEM

Given a finite Blaschke product f and a collection $\mathbf{t} = \{t_1, \dots, t_n\}$ of distinct points in $\overline{\mathbb{D}}$, let us define the associated *Schwarz-Pick matrix*

$$P^f(\mathbf{t}) = \left[p^f(t_i, t_j) \right]_{i,j=1}^n$$

by entry-wise formulas

$$p^f(t_i, t_j) = \begin{cases} t_i f'(t_i) \overline{f(t_i)} & \text{if } i = j \text{ and } t_i \in \mathbb{T}, \\ \frac{1 - f(t_i) \overline{f(t_j)}}{1 - t_i \overline{t_j}} & \text{otherwise.} \end{cases} \quad (2.1)$$

In case $\mathbf{t} \subset \mathbb{T}$, we will refer to $P^f(\mathbf{t})$ as to the *boundary Schwarz-Pick matrix*. Observe that for a finite Blaschke product f , equalities

$$\lim_{z \rightarrow t} \frac{1 - |f(z)|^2}{1 - |z|^2} = t f'(t) \overline{f(t)} = |f'(t)| \quad (2.2)$$

hold at every point $t \in \mathbb{T}$ and thus, the diagonal entries in $P^f(\mathbf{t})$ are all nonnegative. The next result is well-known (see e.g. [3, Lemma 2.1] for the proof).

Lemma 2.1. *For $f \in \mathcal{B}_k^\circ$ and a tuple $\mathbf{t} = \{t_1, \dots, t_n\} \in \overline{\mathbb{D}}^n$, the matrix $P^f(\mathbf{t})$ (2.1) is positive semidefinite and $\text{rank } P^f(\mathbf{t}) = \min\{n, k\}$.*

Boundary interpolation by Schur-class functions and, in particular, by finite Blaschke products becomes more transparent if, in addition to conditions (1.2), one prescribes the values of f' at each interpolation node t_i . We denote this modified problem by **MP**.

MP: *Given $t_i, f_i \in \mathbb{T}$ and $\gamma_i \geq 0$, find a finite Blaschke product f such that*

$$f(t_i) = w_i, \quad |f'(t_i)| = \gamma_i \quad \text{for } i = 1, \dots, n \quad (t_i, w_i \in \mathbb{T}, \gamma_i \geq 0). \quad (2.3)$$

The problem (2.3) is well-known in a more general context of rational functions $f : \mathbb{D} \rightarrow \mathbb{D}$ ([2]) and even in the more general context Schur-class functions ([20, 6]). The results on finite Blaschke product interpolation presented in Theorem 2.2 below are easily derived from the general ones.

Theorem 2.2. *If the problem **MP** has a solution, then the Pick matrix P_n of the problem defined as*

$$P_n = [p_{ij}]_{i,j=1}^n, \quad \text{where } p_{ij} = \begin{cases} \frac{1 - w_i \overline{w_j}}{1 - t_i \overline{t_j}} & \text{if } i \neq j, \\ \gamma_i & \text{if } i = j, \end{cases} \quad (2.4)$$

is positive semidefinite. Moreover,

- (1) *If $P_n > 0$, then the problem **MP** has infinitely many solutions.*

(2) If $P_n \geq 0$ and $\text{rank}(P_n) = k < n$, then there is an $f \in \mathcal{B}_k$ such that

$$f(t_i) = w_i, \quad |f'(t_i)| \leq \gamma_i \quad \text{for } i = 1, \dots, n \quad (2.5)$$

and no other rational function $f : \mathbb{D} \rightarrow \mathbb{D}$ meets conditions (2.5).

Remark 2.3. The second statement in Theorem 2.2 suggests a short proof of the first part of Theorem 1.1. Indeed, given f_1, \dots, f_n , choose positive numbers $\gamma_1, \dots, \gamma_{n-1}$ large enough so that the Pick matrix P_{n-1} defined via formula (2.4) is positive definite and extend P_{n-1} to $P_n = \begin{bmatrix} P_{n-1} & F \\ F^* & \gamma_n \end{bmatrix}$ by letting

$$\gamma_n := F P_{n-1}^{-1} F^*, \quad \text{where } F = [p_{n,1} \ \dots \ p_{n,n-1}], \quad p_{n,j} = \frac{1 - w_n \bar{w}_j}{1 - t_n \bar{t}_j}.$$

By the Schur complement argument, the matrix P_n constructed as above is singular. By Theorem 2.2, there is a finite Blaschke product $f \in \mathcal{B}_{n-1}^\circ$ satisfying conditions (2.5). Clearly, this f solves the problem (1.2).

It is seen from (2.1)–(2.4), that for every solution f to the problem **MP**, the Schwarz-Pick matrix $P^f(\mathbf{t})$ based on the interpolation nodes t_1, \dots, t_n is equal to the matrix P_n constructed in (2.4) from the interpolation data set. Hence, the first statement in Theorem 2.2 follows from Lemma 2.1. The last statement in the theorem can be strengthened as follows: *if P_n is singular, there is an $f \in \mathcal{B}_k$ satisfying conditions (2.5), but there is no another Schur-class function g (even non-rational) such that*

$$\lim_{r \rightarrow 1} g(rt_i) = w_i, \quad \left| \lim_{r \rightarrow 1} g'(rt_i) \right| \leq \gamma_i \quad \text{for } i = 1, \dots, n.$$

In any event, if P_n is singular, there is only one candidate which may or may not be a solution to the problem **MP**. The determinacy criterion for the problem **MP** in terms of the Pick matrix P_n is recalled in Theorem 2.6 below.

Definition 2.4. A positive semidefinite matrix of rank r is called *saturated* if every its $r \times r$ principal submatrix is positive definite. A positive semidefinite matrix is called *minimally positive* if none of its diagonal entries can be decreased so that the modified matrix will be still positive semidefinite.

Remark 2.5. The rank equality in Lemma 2.1 implies that for $f \in \mathcal{B}_k^\circ$ and $k < n$, the Schwarz-Pick matrix (2.1) is saturated.

Theorem 2.6 ([20]). *The problem **MP** has a unique solution if and only if P_n is minimally positive or equivalently, if and only if $P_n \geq 0$ is singular and saturated. The unique solution is a finite Blaschke product of degree equal the rank of P_n .*

3. PARAMETRIZATION OF THE SET OF LOW-DEGREE SOLUTIONS

With the first $n - 1$ interpolation conditions in (1.2) we associate the matrices

$$T = \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_{n-1} \end{bmatrix}, \quad M = \begin{bmatrix} w_1 \\ \vdots \\ w_{n-1} \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \quad (3.1)$$

Definition 3.1. A tuple $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\}$ of positive numbers will be called *admissible* if the Pick matrix P_{n-1}^γ defined as in (2.4) is positive definite:

$$P_{n-1}^\gamma = [p_{ij}]_{i,j=1}^{n-1} > 0, \quad \text{where} \quad p_{ij} = \begin{cases} \frac{1 - w_i \bar{w}_j}{1 - \bar{t}_i t_j} & \text{if } i \neq j, \\ \gamma_i & \text{if } i = j. \end{cases} \quad (3.2)$$

From now on, the tuple $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\}$ will serve as a parameter. Observe the Stein identity

$$P_{n-1}^\gamma - TP_{n-1}^\gamma T^* = EE^* - MM^*, \quad (3.3)$$

which holds true for any choice of $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\} \in \mathbb{R}^{n-1}$. In what follows, \mathbf{e}_i will denote the i -th column in the identity matrix I_{n-1} .

Remark 3.2. If P_{n-1}^γ is invertible, it satisfies the Stein identity

$$(P_{n-1}^\gamma)^{-1} - T^*(P_{n-1}^\gamma)^{-1}T = XX^* - YY^*, \quad (3.4)$$

where the columns $X = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$ are given by

$$\begin{aligned} X &= (I - t_n T^*)(P_{n-1}^\gamma)^{-1}(t_n I - T)^{-1}E, \\ Y &= (I - t_n T^*)(P_{n-1}^\gamma)^{-1}(t_n I - T)^{-1}M. \end{aligned} \quad (3.5)$$

Furthermore, the entries

$$\begin{aligned} x_i &= (1 - t_n \bar{t}_i) \mathbf{e}_i^* (P_{n-1}^\gamma)^{-1} (t_n I - T)^{-1} E, \\ y_i &= (1 - t_n \bar{t}_i) \mathbf{e}_i^* (P_{n-1}^\gamma)^{-1} (t_n I - T)^{-1} M, \end{aligned} \quad (i = 1, \dots, n-1) \quad (3.6)$$

in the columns (3.5) are subject to equalities

$$|x_i| = |y_i| \neq 0 \quad \text{for } i = 1, \dots, n-1. \quad (3.7)$$

Proof. Making use of (3.5) and (3.3), we verify (3.4) as follows:

$$\begin{aligned} XX^* - YY^* &= (I - t_n T^*)(P_{n-1}^\gamma)^{-1}(t_n I - T)^{-1} [P_{n-1}^\gamma - TP_{n-1}^\gamma T^*] \\ &\quad \times (\bar{t}_n I - T^*)^{-1} (P_{n-1}^\gamma)^{-1} (I - \bar{t}_n T) \\ &= (I - t_n T^*)(P_{n-1}^\gamma)^{-1} [P_{n-1}^\gamma (I - t_n T^*)^{-1} + \\ &\quad + (I - \bar{t}_n T)^{-1} \bar{t}_n T P_{n-1}^\gamma] \times (P_{n-1}^\gamma)^{-1} (I - \bar{t}_n T) \\ &= (P_{n-1}^\gamma)^{-1} (I - \bar{t}_n T) + (I - t_n T^*)(P_{n-1}^\gamma)^{-1} \bar{t}_n T \\ &= (P_{n-1}^\gamma)^{-1} - T^* (P_{n-1}^\gamma)^{-1} T. \end{aligned}$$

Comparing the corresponding diagonal entries on both sides of (3.4) gives $0 = |x_i|^2 - |y_i|^2$ so that $|x_i| = |y_i|$ for $i = 1, \dots, n-1$. To complete the proof of (3.7), it suffices to show that x_i and y_i cannot be both equal zero. To this end, we compare the i -th rows of both sides in (3.4) to get

$$x_i X^* - y_i Y^* = \mathbf{e}_i^* (P_{n-1}^\gamma)^{-1} - \mathbf{e}_i^* T^* (P_{n-1}^\gamma)^{-1} T = \mathbf{e}_i^* (P_{n-1}^\gamma)^{-1} (I - \bar{t}_i T). \quad (3.8)$$

Let us assume that $x_i = y_i = 0$. Then the expression on the right side of (3.8) is the zero row-vector. Since $(I - \bar{t}_i T)$ is the diagonal matrix with the i -th diagonal entry equal zero and all other diagonal entries being non-zero, it follows that all entries in the row-vector $\mathbf{e}_i^* (P_{n-1}^\gamma)^{-1}$, except the i -th entry, are zeroes, so that $\mathbf{e}_i^* (P_{n-1}^\gamma)^{-1} = \alpha \mathbf{e}_i^*$ for some $\alpha \in \mathbb{C}$. Then it follows from (3.6) and (3.1) that

$$x_i = \alpha(1 - t_n \bar{t}_i) \mathbf{e}_i^* (t_n I - T)^{-1} E = -\alpha \bar{t}_i.$$

Since $x_i = 0$ and $t_i \neq 0$, it follows that $\alpha = 0$ and hence, $\mathbf{e}_i^* (P_{n-1}^\gamma)^{-1} = \alpha \mathbf{e}_i^* = 0$. The latter cannot happen since the matrix P_{n-1}^γ is invertible. The obtained contradiction completes the proof of (3.7). \square

For each admissible tuple $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\}$, we define the 2×2 matrix function

$$\begin{aligned} \Theta^\gamma(z) &= \begin{bmatrix} \theta_{11}^\gamma(z) & \theta_{12}^\gamma(z) \\ \theta_{21}^\gamma(z) & \theta_{22}^\gamma(z) \end{bmatrix} \\ &= I + (z - t_n) \begin{bmatrix} E^* \\ M^* \end{bmatrix} (I - zT^*)^{-1} (P_{n-1}^\gamma)^{-1} (t_n I - T)^{-1} \begin{bmatrix} E & -M \end{bmatrix}, \end{aligned} \quad (3.9)$$

where T, M, E are defined as in (3.1). Upon making use of the columns (3.5) and of the diagonal structure of T , we may write the formula (3.9) for Θ^γ as

$$\begin{aligned} \Theta^\gamma(z) &= I + (z - t_n) \begin{bmatrix} E^* \\ M^* \end{bmatrix} (I - zT^*)^{-1} (I - t_n T^*)^{-1} \begin{bmatrix} X & -Y \end{bmatrix} \\ &= I + \sum_{i=1}^{n-1} \frac{z - t_n}{(1 - z\bar{t}_i)(1 - t_n \bar{t}_i)} \cdot \begin{bmatrix} 1 \\ \bar{w}_i \end{bmatrix} \begin{bmatrix} x_i & -y_i \end{bmatrix} \\ &= I + \sum_{i=1}^{n-1} \left(\frac{t_i}{1 - z\bar{t}_i} - \frac{t_i}{1 - t_n \bar{t}_i} \right) \cdot \begin{bmatrix} 1 \\ \bar{w}_i \end{bmatrix} \begin{bmatrix} x_i & -y_i \end{bmatrix}. \end{aligned} \quad (3.10)$$

It is seen from (3.10) that Θ^γ is rational with simple poles at t_1, \dots, t_{n-1} . We next summarize some other properties of Θ^γ needed for our subsequent analysis.

Theorem 3.3. *Let $P_{n-1}^\gamma > 0$, let X, Y, Θ^γ be defined as in (3.5), (3.9), and let*

$$\Upsilon(z) = \prod_{i=1}^{n-1} (1 - z\bar{t}_i) \quad \text{and} \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (3.11)$$

(1) *For every unimodular constant w_n , the functions*

$$p(z) = \Upsilon(z) (\theta_{11}^\gamma(z) w_n + \theta_{12}^\gamma(z)), \quad q(z) = \Upsilon(z) (\theta_{21}^\gamma(z) w_n + \theta_{22}^\gamma(z)) \quad (3.12)$$

are polynomials of degree $n - 1$ with all zeros in $\overline{\mathbb{D}}$ and in $\mathbb{C} \setminus \mathbb{D}$, respectively.

(2) *The following identities hold for any $z, \zeta \in \mathbb{C}$ ($z\bar{\zeta} \neq 1$):*

$$\frac{J - \Theta^\gamma(z) J \Theta^\gamma(\zeta)^*}{1 - z\bar{\zeta}} = \begin{bmatrix} E^* \\ M^* \end{bmatrix} (I - zT^*)^{-1} (P_{n-1}^\gamma)^{-1} (I - \bar{\zeta}T)^{-1} \begin{bmatrix} E & M \end{bmatrix}, \quad (3.13)$$

$$\frac{J - \Theta^\gamma(\zeta)^* J \Theta^\gamma(z)}{1 - z\bar{\zeta}} = \begin{bmatrix} X^* \\ -Y^* \end{bmatrix} (I - \bar{\zeta}T)^{-1} P_{n-1}^\gamma (I - zT^*)^{-1} \begin{bmatrix} X & -Y \end{bmatrix}. \quad (3.14)$$

(3) $\det \Theta^\gamma(z) = 1$ for all $z \in \mathbb{C} \setminus \{t_1, \dots, t_n\}$.

The proofs of (1)–(3) can be found in [2] for a more general framework where P_{n-1}^γ is an invertible Hermitian matrix satisfying the Stein identity (3.3) for some T , E and M (though, of the same dimensions as in (3.1)) and t_n is an arbitrary point in $\mathbb{T} \setminus \sigma(T)$. In this more general setting, p has $\pi(P_{n-1}^\gamma)$ zeroes in $\overline{\mathbb{D}}$ and q has $\nu(P_{n-1}^\gamma)$ zeroes in $\mathbb{C} \setminus \overline{\mathbb{D}}$ where $\pi(P_{n-1}^\gamma)$ and $\nu(P_{n-1}^\gamma)$ are respectively the number of positive and the number of negative eigenvalues of $\nu(P_{n-1}^\gamma)$ counted with multiplicities. Straightforward verifications of (3.13) and (3.14) rely solely on the Stein identities (3.3) and (3.4), respectively. Another calculation based on (3.3) shows that

$$\det \Theta^\gamma(z) = \det [(zI - T)(\bar{t}_n I - T^*)(I - zT^*)^{-1}(1 - \bar{t}_n T)^{-1}]$$

which is equal to one due to a special form (3.1) of T .

Remark 3.4. Let $P \in \mathbb{C}^{n \times n}$ be a positive semidefinite saturated matrix with $\text{rank } P = k < n$. Let $G \in \mathbb{C}^{n \times n}$ be a diagonal positive semidefinite matrix with $\text{rank } G = m < n - k$. Then $\text{rank}(P + G) = k + m$.

Indeed, since the matrix $\Phi P \Phi^{-1}$ is saturated for any permutation matrix Φ , we may take P and G conformally decomposed as follows:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{G} \end{bmatrix}, \quad P_{11} \in \mathbb{C}^{k \times k}.$$

Since $\text{rank } P = \text{rank } P_{11} = k$ (i.e., P_{11} is invertible), it follows the Schur complement of P_{11} in P is equal to the zero matrix: $P_{22} - P_{21}P_{11}^{-1}P_{12} = 0$. By the formula for the rank of a block matrix, we then have

$$\begin{aligned} \text{rank}(P + G) &= \text{rank} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} + \tilde{G} \end{bmatrix} = \text{rank } P_{11} + \text{rank}(P_{22} + \tilde{G} - P_{21}P_{11}^{-1}P_{12}) \\ &= k + \text{rank } \tilde{G} = k + \text{rank } G = k + m. \end{aligned}$$

The next theorem is the main result of this section.

Theorem 3.5. Given data $(t_i, w_i) \in \mathbb{T}^2$ ($i = 1, \dots, n$), let $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\}$ be an admissible tuple and let

$$f_\gamma(z) = \frac{\theta_{11}^\gamma(z)w_n + \theta_{12}^\gamma(z)}{\theta_{21}^\gamma(z)w_n + \theta_{22}^\gamma(z)} \quad (3.15)$$

where the coefficients θ_{ij}^γ in (3.15) are constructed from γ by formula (3.6). Let x_i and y_i be the numbers defined as in (3.6). Then

- (1) f_γ is a finite Blaschke product and satisfies conditions (1.2).
- (2) f_γ satisfies conditions $|f'_\gamma(t_i)| \leq \gamma_i$ for $i = 1, \dots, n-1$. Moreover, $|f'_\gamma(t_i)| = \gamma_i$ if and only if $x_i w_n \neq y_i$.
- (3) $\deg f_\gamma = n - 1 - \ell$, where $\ell = \#\{i \in \{1, \dots, n-1\} : x_i w_n = y_i\}$.

Conversely every Blaschke product $f \in \mathcal{B}_{n-1}$ subject to interpolation conditions (1.2) admits a representation (3.15) for some admissible tuple $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\}$. This representation is unique if and only if $\deg f = n - 1$.

Proof. Let $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\}$ be an admissible tuple and let Θ^γ be defined as in (3.9). Since Θ^γ is rational, the function f_γ is rational as well. Let

$$N(z) = \theta_{11}^\gamma(z)w_n + \theta_{12}^\gamma(z) \quad \text{and} \quad D(z) = \theta_{21}^\gamma(z)w_n + \theta_{22}^\gamma(z) \quad (3.16)$$

denote the numerator and the denominator in (3.15) and let

$$\Psi(z) = (I - zT^*)^{-1}(Xw_n - Y) = \sum_{i=1}^{n-1} \mathbf{e}_i \frac{x_i w_n - y_i}{1 - z\bar{t}_i}. \quad (3.17)$$

Combining (3.16) and (3.10) gives

$$\begin{bmatrix} N(z) \\ D(z) \end{bmatrix} = \Theta^\gamma(z) \begin{bmatrix} w_n \\ 1 \end{bmatrix} = \begin{bmatrix} w_n \\ 1 \end{bmatrix} + (z - t_n) \begin{bmatrix} E^* \\ M^* \end{bmatrix} (I - t_n T^*)^{-1} \Psi(z). \quad (3.18)$$

Taking the advantage of the matrix J in (3.11) and of relation (3.18) we get

$$\begin{aligned} |D(z)|^2 - |N(z)|^2 &= -[N(z)^* \quad D(z)^*] J \begin{bmatrix} N(z) \\ D(z) \end{bmatrix} \\ &= -[\bar{w}_n \quad 1] \Theta^\gamma(z)^* J \Theta^\gamma(z) \begin{bmatrix} w_n \\ 1 \end{bmatrix} \\ &= [\bar{w}_n \quad 1] \{J - \Theta^\gamma(z)^* J \Theta^\gamma(z)\} \begin{bmatrix} w_n \\ 1 \end{bmatrix}, \end{aligned} \quad (3.19)$$

where for the third equality we used

$$[\bar{w}_n \quad 1] J \begin{bmatrix} w_n \\ 1 \end{bmatrix} = 1 - |w_n|^2 = 0.$$

Substituting (3.14) into (3.19) and making use of notation (3.17), we conclude

$$|D(z)|^2 - |N(z)|^2 = (1 - |z|^2) \Psi(z)^* P_{n-1}^\gamma \Psi(z).$$

Combining the latter equality with (3.15) and (3.16) gives

$$\frac{1 - |f_\gamma(z)|^2}{1 - |z|^2} = \frac{|D(z)|^2 - |N(z)|^2}{(1 - |z|^2)|D(z)|^2} = \frac{\Psi(z)^* P_{n-1}^\gamma \Psi(z)}{|D(z)|^2}, \quad (3.20)$$

which implies, in particular, that f_γ is inner. Since f_γ is rational, it extends by continuity to a finite Blaschke product. One can see from (3.17) that

$$\lim_{z \rightarrow t_i} (z - t_i) \cdot \Psi(z) = -\mathbf{e}_i t_i (x_i w_n - y_i) \quad \text{for } i = 1, \dots, n-1, \quad (3.21)$$

which together with (3.18) and (3.1) implies

$$\begin{aligned} \lim_{z \rightarrow t_i} (z - t_i) \cdot \begin{bmatrix} N(z) \\ D(z) \end{bmatrix} &= (t_n - t_i) \begin{bmatrix} E^* \\ M^* \end{bmatrix} (I - t_n T^*)^{-1} \mathbf{e}_i t_i (x_i w_n - y_i) \\ &= -\left[\frac{1}{\bar{w}_i}\right] t_i^2 (x_i w_n - y_i) \quad \text{for } i = 1, \dots, n-1. \end{aligned} \quad (3.22)$$

To show that f_γ satisfies conditions (1.2), we first observe that $\Theta^\gamma(t_n) = I$ (by definition (3.9)); now the equality $f_\gamma(t_n) = w_n$ is immediate from (3.15). Verification of other equalities in (1.2) depends on whether or not $x_i w_n = y_i$.

Case 1: Let us assume that $x_i w_n \neq y_i$. Then we have from (3.15) and (3.22),

$$f_\gamma(t_i) = \frac{(z - t_i)N(z)}{(z - t_i)D(z)} = \frac{t_i^2(x_i w_n - y_i)}{\bar{w}_i t_i^2(x_i w_n - y_i)} = \frac{1}{\bar{w}_i} = w_i. \quad (3.23)$$

Under the same assumption, we conclude from (2.2), (3.2) and (3.20)–(3.22),

$$\begin{aligned} |f'_\gamma(t_i)| &= \lim_{z \rightarrow t_i} \frac{1 - |f_\gamma(z)|^2}{1 - |z|^2} = \lim_{z \rightarrow t_i} \frac{|z - t_i|^2 \Psi(z)^* P_{n-1}^\gamma \Psi(z)}{|z - t_i|^2 |D(z)|^2} \\ &= \frac{|x_i w_n - y_i|^2 \mathbf{e}_i^* P_{n-1}^\gamma \mathbf{e}_i}{|x_i w_n - y_i|^2} = \gamma_i. \end{aligned}$$

Case 2: Let us assume that $x_i w_n = y_i$. Then the functions N , D and Ψ are analytic at t_i . To compute the values of these functions at $z = t_i$, we first take the adjoints of both sides in (3.8):

$$X\bar{x}_i - Y\bar{y}_i = (I - t_i T^*)(P_{n-1}^\gamma)^{-1} \mathbf{e}_i.$$

We next divide both sides by \bar{y}_i and make use of equalities $w_n = y_i/x_i = \bar{x}_i/\bar{y}_i$ (by the assumption of Case 2) to get

$$Xw_n - Y = \frac{1}{\bar{y}_i} (I - t_i T^*)(P_{n-1}^\gamma)^{-1} \mathbf{e}_i.$$

Substituting the latter equality into (3.17) results in

$$\Psi(z) = \frac{1}{\bar{y}_i} (I - z T^*)^{-1} (I - t_i T^*)(P_{n-1}^\gamma)^{-1} \mathbf{e}_i,$$

which being evaluated at t_i , gives

$$\Psi(t_i) = \frac{1}{\bar{y}_i} (I - \mathbf{e}_i \mathbf{e}_i^*)(P_{n-1}^\gamma)^{-1} \mathbf{e}_i = \frac{1}{\bar{y}_i} ((P_{n-1}^\gamma)^{-1} \mathbf{e}_i - \mathbf{e}_i \tilde{p}_{ii}), \quad (3.24)$$

where \tilde{p}_{ii} denotes the i -th diagonal entry of $(P_{n-1}^\gamma)^{-1}$. Evaluating the formula (3.18) at $z = t_i$ gives, in view of (3.24),

$$\begin{bmatrix} N(t_i) \\ D(t_i) \end{bmatrix} = \begin{bmatrix} w_n \\ 1 \end{bmatrix} + \frac{t_i - t_n}{\bar{y}_i} \begin{bmatrix} E^* \\ M^* \end{bmatrix} (I - t_n T^*)^{-1} ((P_{n-1}^\gamma)^{-1} \mathbf{e}_i - \mathbf{e}_i \tilde{p}_{ii}).$$

Making use of formulas (3.5), we have

$$\begin{aligned} \frac{t_i - t_n}{\bar{y}_i} \begin{bmatrix} E^* \\ M^* \end{bmatrix} (I - t_n T^*)^{-1} (P_{n-1}^\gamma)^{-1} \mathbf{e}_i &= \frac{t_i - t_n}{\bar{y}_i} \begin{bmatrix} X^* \\ Y^* \end{bmatrix} (t_n I - T)^{-1} \mathbf{e}_i \\ &= -\frac{1}{\bar{y}_i} \begin{bmatrix} X^* \mathbf{e}_i \\ Y^* \mathbf{e}_i \end{bmatrix} = -\frac{1}{\bar{y}_i} \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \end{bmatrix} = -\begin{bmatrix} w_n \\ 1 \end{bmatrix}. \end{aligned}$$

Combining the two latter formulas and again making use of (3.1) leads us to

$$\begin{aligned} \begin{bmatrix} N(t_i) \\ D(t_i) \end{bmatrix} &= \frac{t_n - t_i}{\bar{y}_i} \begin{bmatrix} E^* \\ M^* \end{bmatrix} (I - t_n T^*)^{-1} \mathbf{e}_i \tilde{p}_{ii} \\ &= \frac{t_n - t_i}{\bar{y}_i (1 - t_n \bar{t}_i)} \begin{bmatrix} E^* \mathbf{e}_i \\ M^* \mathbf{e}_i \end{bmatrix} \tilde{p}_{ii} = -\begin{bmatrix} 1 \\ \bar{w}_i \end{bmatrix} \frac{t_i \tilde{p}_{ii}}{\bar{y}_i}. \end{aligned}$$

Thus,

$$N(t_i) = -\frac{t_i \tilde{p}_{ii}}{\tilde{y}_i} \quad \text{and} \quad D(t_i) = -\frac{t_i \tilde{p}_{ii} \bar{w}_i}{\tilde{y}_i}, \quad (3.25)$$

and subsequently, $f_\gamma(t_i) = \frac{N(t_i)}{D(t_i)} = \frac{1}{\bar{w}_i} = w_i$. Furthermore, we have from (2.2) and (3.20),

$$|f'_\gamma(t_i)| = \lim_{z \rightarrow t_i} \frac{1 - |f_\gamma(z)|^2}{1 - |z|^2} = \lim_{z \rightarrow t_i} \frac{\Psi(z)^* P_{n-1}^\gamma \Psi(z)}{|D(z)|^2} = \frac{\Psi(t_i)^* P_{n-1}^\gamma \Psi(t_i)}{|D(t_i)|^2}. \quad (3.26)$$

In view of (3.24) and (3.25),

$$\begin{aligned} \Psi(t_i)^* P_{n-1}^\gamma \Psi(t_i) &= \frac{1}{|y_i|^2} (\mathbf{e}_i^* (P_{n-1}^\gamma)^{-1} - \tilde{p}_{ii} \mathbf{e}_i^*) P_{n-1}^\gamma ((P_{n-1}^\gamma)^{-1} \mathbf{e}_i - \mathbf{e}_i \tilde{p}_{ii}) \\ &= \frac{1}{|y_i|^2} (\mathbf{e}_i^* (P_{n-1}^\gamma)^{-1} \mathbf{e}_i - 2\tilde{p}_{ii} + \tilde{p}_{ii}^2 \mathbf{e}_i^* P_{n-1}^\gamma \mathbf{e}_i) = \frac{\gamma_i \tilde{p}_{ii}^2 - \tilde{p}_{ii}}{|y_i|^2}, \\ |D(t_i)|^2 &= \frac{\tilde{p}_{ii}^2}{|y_i|^2}. \end{aligned}$$

Substituting the two latter equalities into the right hand side of (3.26) we get

$$|f'_\gamma(t_i)| = \frac{\gamma_i \tilde{p}_{ii}^2 - \tilde{p}_{ii}}{\tilde{p}_{ii}^2} = \gamma_i - \frac{1}{\tilde{p}_{ii}} < \gamma_i.$$

We have verified equalities (1.2) and we showed that $|f'_\gamma(t_i)| \leq \gamma_i$ with strict inequality if and only if $x_i w_n = y_i$ (i.e., in Case 2). This completes the proof of statements (1) and (2) of the theorem.

To prove part (3), we multiply the numerator and the denominator on the right hand side of (3.15) by Υ (see formula (3.11)) to get a linear fractional representation for f with polynomial coefficients $\tilde{\theta}_{ij}^\gamma = \Upsilon \theta_{ij}^\gamma$:

$$f_\gamma(z) = \frac{\tilde{\theta}_{11}^\gamma(z) w_n + \tilde{\theta}_{12}^\gamma(z)}{\tilde{\theta}_{21}^\gamma(z) w_n + \tilde{\theta}_{22}^\gamma(z)} = \frac{\Upsilon(z) N(z)}{\Upsilon(z) D(z)} = \frac{p(z)}{q(z)}, \quad (3.27)$$

where p and q are the polynomials given in (3.12). Since the resulting function f_γ extends to a finite Blaschke product (with no poles or zeroes on \mathbb{T}), it follows that p and q have the same (if any) zeroes on \mathbb{T} counted with multiplicities. The common zeroes may occur only at the zeroes of the determinant of the coefficient matrix and since $\det(\Upsilon(z) \tilde{\Theta}^\gamma(z)) = [\Upsilon(z)]^2$ (by statement (3) in Theorem 3.3 and by analyticity of $\det(\Upsilon(z) \tilde{\Theta}^\gamma(z))$), it follows that p and q may have common zeros only at t_1, \dots, t_{n-1} . By (3.22) and (3.11), $p(t_i) = 0$ if and only if $x_i w_n = y_i$. On the other hand, if this is the case, $D(t_i) \neq 0$ (by formula (3.25)) and therefore, t_i is a *simple* zero of $p = \Upsilon D$. Thus, D may have only simple zeros at t_1, \dots, t_{n-1} . We summarize: by statement (4) in Theorem 3.3, the numerator p in (3.27) has $n - 1$ zeroes in $\overline{\mathbb{D}}$. All zeroes of p and q on \mathbb{T} are simple and common; they occur precisely at those t_i 's for which $x_i w_n = y_i$. After zero cancellations, the function f_γ turns out to be a finite Blaschke product of degree $n - 1 - \#\{i \in \{1, \dots, n - 1\} : x_i w_n = y_i\}$. This completes the proof of part (3).

To prove the converse statement, let us assume that f is a Blaschke product of degree $k \leq n-1$ that satisfies conditions (1.2). For a fixed permutation $\{i_1, \dots, i_{n-1}\}$ of the index set $\{1, \dots, n-1\}$, we choose the integers $\gamma_1, \dots, \gamma_{n-1}$ so that

$$\gamma_{i_j} = |f'(t_{i_j})| \quad (1 \leq j \leq k) \quad \text{and} \quad \gamma_{i_j} > |f'(t_{i_j})| \quad (k < j \leq n-1). \quad (3.28)$$

Due to this choice, the diagonal matrix

$$G = \begin{bmatrix} \gamma_1 - |f'(t_1)| & & 0 \\ & \ddots & \\ 0 & & \gamma_{n-1} - |f'(t_{n-1})| \end{bmatrix} \quad (3.29)$$

is positive semidefinite and $\text{rank } G = n - k - 1$. We are going to show that the tuple $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\}$ is admissible and that $f = f_\gamma$ as in (3.15).

Let $P^f = P^f(t_1, \dots, t_{n-1}, z)$ be the Schwarz-Pick matrix of f based on the interpolation nodes t_1, \dots, t_{n-1} and one additional point $z \in \mathbb{D}$. According to (2.1) and due to interpolation conditions (1.2), this matrix has the form

$$P^f = \begin{bmatrix} |f'(t_1)| & \frac{1-w_1\bar{w}_2}{1-t_1\bar{t}_2} & \cdots & \frac{1-w_1\bar{w}_{n-1}}{1-t_1\bar{t}_{n-1}} & \frac{1-w_1\overline{f(z)}}{1-t_1\bar{z}} \\ \frac{1-w_2\bar{w}_1}{1-t_2\bar{t}_1} & |f'(t_2)| & \cdots & \frac{1-w_2\bar{w}_{n-1}}{1-t_2\bar{t}_{n-1}} & \frac{1-w_2\overline{f(z)}}{1-t_2\bar{z}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1-w_{n-1}\bar{w}_1}{1-t_{n-1}\bar{t}_1} & \frac{1-w_{n-1}\bar{w}_2}{1-t_{n-1}\bar{t}_2} & \cdots & |f'(t_{n-1})| & \frac{1-w_{n-1}\overline{f(z)}}{1-t_{n-1}\bar{z}} \\ \frac{1-f(z)\bar{w}_1}{1-z\bar{t}_1} & \frac{1-f(z)\bar{w}_2}{1-z\bar{t}_2} & \cdots & \frac{1-f(z)\bar{w}_{n-1}}{1-z\bar{t}_{n-1}} & \frac{1-|f(z)|^2}{1-|z|^2} \end{bmatrix}. \quad (3.30)$$

Observe that the leading $(n-1) \times (n-1)$ submatrix of P^f is the boundary Schwarz-Pick matrix $P^f(t_1, \dots, t_{n-1})$, while the bottom row in P^f (without the rightmost entry) can be written in terms of the matrices (3.1) as $(E^* - f(z)M^*)(I - zT^*)^{-1}$. Thus, P^f can be written in a more compact form

$$P^f = \begin{bmatrix} P^f(t_1, \dots, t_{n-1}) & (I - \bar{z}T)^{-1}(E - M\overline{f(z)}) \\ (E^* - f(z)M^*)(I - zT^*)^{-1} & \frac{1 - |f(z)|^2}{1 - |z|^2} \end{bmatrix}. \quad (3.31)$$

Let P_{n-1}^γ be the matrix defined via formulas (3.2) and let

$$\mathbb{P}^\gamma(z) := \begin{bmatrix} P_{n-1}^\gamma & (I - \bar{z}T)^{-1}(E - M\overline{f(z)}) \\ (E^* - f(z)M^*)(I - zT^*)^{-1} & \frac{1 - |f(z)|^2}{1 - |z|^2} \end{bmatrix}. \quad (3.32)$$

Taking into account the formula (3.29) for G and comparing (3.2) and (3.31) with (2.1) and (3.32), respectively, leads us to equalities

$$P_{n-1}^\gamma = P^f(t_1, \dots, t_{n-1}) + G \quad \text{and} \quad \mathbb{P}^\gamma(z) = P^f + \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix}. \quad (3.33)$$

By Remark 2.5, the Schwarz-Pick matrices P^f and $P^f(t_1, \dots, t_{n-1})$ are positive semidefinite and saturated. Moreover, since $f \in \mathcal{B}_k^\circ$, we have

$$\text{rank } P^f = \text{rank } P^f(t_1, \dots, t_{n-1}) = k,$$

by Lemma 2.1. Then it follows from (3.33) by Remark 3.4 that

$$\text{rank } P_{n-1}^\gamma = \text{rank } P^f(t_1, \dots, t_{n-1}) + \text{rank } G = n - 1 \quad (3.34)$$

(i.e., P_{n-1}^γ is positive definite and hence $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\}$ is admissible) and

$$\text{rank } \mathbb{P}^\gamma(z) = \text{rank } P^f + \text{rank } \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} = n - 1 \quad \text{for all } z \in \mathbb{D}. \quad (3.35)$$

By (3.34) and (3.35), the Schur complement of the block P_{n-1}^γ in (3.31) is equal to zero for every $z \in \mathbb{D}$:

$$\frac{1 - |f(z)|^2}{1 - |z|^2} - (E^* - f(z)M^*)(I - zT^*)^{-1}(P_{n-1}^\gamma)^{-1}(I - \bar{z}T)^{-1}(E - M\overline{f(z)}) = 0. \quad (3.36)$$

The rational matrix-function Θ^γ constructed from $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\}$ via formula (3.9) satisfies the identity (3.13). Multiplying the latter identity (with $\zeta = z$) by the row-vector $[1 \quad -f(z)]$ on the left and by its adjoint on the right gives

$$\begin{aligned} & (E^* - f(z)M^*)(I - zT^*)^{-1}(P_{n-1}^\gamma)^{-1}(I - \bar{z}T)^{-1}(E - M\overline{f(z)}) \\ &= [1 \quad -f(z)] \frac{J - \Theta^\gamma(z)J\Theta^\gamma(z)^*}{1 - |z|^2} \begin{bmatrix} 1 \\ -f(z) \end{bmatrix} \\ &= \frac{1 - |f(z)|^2}{1 - |z|^2} - [1 \quad -f(z)] \frac{\Theta^\gamma(z)J\Theta^\gamma(z)^*}{1 - |z|^2} \begin{bmatrix} 1 \\ -f(z) \end{bmatrix} \end{aligned}$$

which, being combined with (3.36), implies

$$[1 \quad -f(z)] \frac{\Theta^\gamma(z)J\Theta^\gamma(z)^*}{1 - |z|^2} \begin{bmatrix} 1 \\ -f(z) \end{bmatrix} = 0 \quad \text{for all } z \in \mathbb{D}. \quad (3.37)$$

Let us consider the functions

$$g = \theta_{11}^\gamma - f\theta_{21}^\gamma \quad \text{and} \quad \mathcal{E} = \frac{f\theta_{22}^\gamma - \theta_{12}^\gamma}{\theta_{11}^\gamma - f\theta_{21}^\gamma}. \quad (3.38)$$

The function g is rational and due to (3.9), $g(t_n) = \theta_{11}^\gamma(t_n) - f(t_n)\theta_{21}^\gamma(t_n) = 1$. Hence, $g \not\equiv 0$ and the rational function \mathcal{E} in (3.38) is well defined. Again, due to (3.9) and the n -th interpolation condition in (1.1),

$$\mathcal{E}(t_n) = f(t_n)\theta_{22}^\gamma(t_n) - \theta_{12}^\gamma(t_n) = f(t_n) = w_n. \quad (3.39)$$

Using the functions (3.38) we now rewrite equality (3.37) as

$$0 = |g(z)|^2 \cdot [1 \quad -\mathcal{E}(z)] \frac{J}{1 - |z|^2} \begin{bmatrix} 1 \\ -\mathcal{E}(z) \end{bmatrix} = \frac{|g(z)|^2(1 - |\mathcal{E}(z)|^2)}{1 - |z|^2}.$$

Since the latter equality holds for all $z \in \mathbb{D}$ and $g \not\equiv 0$, it follows that $|\mathcal{E}(z)| = 1$ for all $z \in \mathbb{D}$ so that \mathcal{E} is a unimodular constant. By (3.39), $\mathcal{E} \equiv w_n$. Now representation (3.15) follows from the the second formula in (3.38).

Finally, if $k = \deg f < n - 1$, then $n - k - 1$ parameters in (3.28) can be increased to produce various admissible tuples γ such that $f = f_\gamma$. On the other hand, if $f \in \mathcal{B}_{n-1}$ admits two different representations (3.15), then for one of them (say, based on an admissible tuple $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\}$), we must

have $\gamma_i \neq |f'(t_i)|$ for some $i \in \{1, \dots, n-1\}$. Then $x_i w_n = y_i$, by part (2) of the theorem, and hence, $\deg f < n-1$, by part (3). Thus, the representation $f = f_\gamma$ is unique if and only if $\deg f = n-1$, which completes the proof of the theorem. \square

We now reformulate Theorem 3.5 in the form that is more convenient for numerical computations. To this end, we let

$$\mathbf{p}_n = \begin{bmatrix} p_{1,n} \\ \vdots \\ p_{n-1,n} \end{bmatrix}, \quad \text{where} \quad p_{i,n} = \frac{1 - w_i \bar{w}_n}{1 - t_i \bar{t}_n}, \quad (3.40)$$

and, for an admissible tuple $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\}$ and the corresponding $P_{n-1}^\gamma > 0$, we let $\Delta^\gamma = (P_{n-1}^\gamma)^{-1} \mathbf{p}_n$. If we denote by $P_{n-1,i}^\gamma(\mathbf{p}_n)$ the matrix obtained from P_{n-1}^γ by replacing its i -th column by \mathbf{p}_n , then by Cramer's rule, we have

$$\Delta^\gamma = \begin{bmatrix} \Delta_1^\gamma \\ \vdots \\ \Delta_{n-1}^\gamma \end{bmatrix}, \quad \Delta_i^\gamma = \frac{\det P_{n-1,i}^\gamma(\mathbf{p}_n)}{\det P_{n-1}^\gamma} \quad (i = 1, \dots, n-1). \quad (3.41)$$

Theorem 3.6. *Given data $(t_i, w_i) \in \mathbb{T}^2$ ($i = 1, \dots, n$), let \mathbf{p}_n be defined as in (3.40). For any admissible tuple $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\}$, the function*

$$f_\gamma(z) = w_n \cdot \frac{1 - (1 - z\bar{t}_n) \cdot \sum_{i=1}^{n-1} \frac{\Delta_i^\gamma}{1 - z\bar{t}_i}}{1 - (1 - z\bar{t}_n) \cdot \sum_{i=1}^{n-1} \frac{\bar{w}_i w_n \Delta_i^\gamma}{1 - z\bar{t}_i}} \quad (3.42)$$

with the numbers Δ_i^γ defined as in (3.41), is the Blaschke product of degree $\deg f_\gamma = n-1 - \#\{i \in \{1, \dots, n-1\} : \Delta_i^\gamma = 0\}$ and satisfies conditions (1.2). Moreover, $|f'_\gamma(t_i)| = \gamma_i$ if and only if $\Delta_i^\gamma \neq 0$ and $|f'_\gamma(t_i)| < \gamma_i$ otherwise.

Proof. Observe that the column (3.40) can be written in terms of the matrices (3.1) as $\mathbf{p}_n = (I - \bar{t}_n T)^{-1} (E - M \bar{w}_n)$. Let x_i and y_i be the numbers defined in (3.6). From formulas (3.6) and (3.41), we have

$$\begin{aligned} x_i w_n - y_i &= (1 - t_n \bar{t}_i) \mathbf{e}_i^* (P_{n-1}^\gamma)^{-1} (t_n I - T)^{-1} (E w_n - M) \\ &= (1 - t_n \bar{t}_i) \mathbf{e}_i^* (P_{n-1}^\gamma)^{-1} (I - \bar{t}_n T)^{-1} (E - M \bar{w}_n) w_n \bar{t}_n \\ &= (1 - t_n \bar{t}_i) \mathbf{e}_i^* (P_{n-1}^\gamma)^{-1} \mathbf{p}_n w_n \bar{t}_n \\ &= (\bar{t}_n - \bar{t}_i) \mathbf{e}_i^* \Delta^\gamma w_n = (\bar{t}_n - \bar{t}_i) \Delta_i^\gamma w_n. \end{aligned} \quad (3.43)$$

Since $t_n \neq t_i$ and $w_n \neq 0$, it now follows that

$$x_i w_n = y_i \iff \Delta_i^\gamma = 0 \iff \det P_{n-1,i}^\gamma(\mathbf{p}_n) = 0. \quad (3.44)$$

We next observe that by the second representation for Θ^γ in (3.10) and (3.43),

$$\begin{aligned}\Theta^\gamma(z) \begin{bmatrix} w_n \\ 1 \end{bmatrix} &= \begin{bmatrix} w_n \\ 1 \end{bmatrix} + \sum_{i=1}^{n-1} \begin{bmatrix} 1 \\ \bar{w}_i \end{bmatrix} \cdot \frac{(z - t_n)(x_i w_n - y_i)}{(1 - z \bar{t}_i)(1 - t_n \bar{t}_i)} \\ &= \begin{bmatrix} w_n \\ 1 \end{bmatrix} + \sum_{i=1}^{n-1} \begin{bmatrix} 1 \\ \bar{w}_i \end{bmatrix} \cdot \frac{(z - t_n)(\bar{t}_n - \bar{t}_i) \Delta_i^\gamma w_n}{(1 - z \bar{t}_i)(1 - t_n \bar{t}_i)} \\ &= \begin{bmatrix} w_n \\ 1 \end{bmatrix} - \sum_{i=1}^{n-1} \begin{bmatrix} 1 \\ \bar{w}_i \end{bmatrix} \cdot \frac{(1 - z \bar{t}_n) \Delta_i^\gamma w_n}{1 - z \bar{t}_i},\end{aligned}$$

from which it follows that formulas (3.42) and (3.15) represent the same function f_γ . Now all statements in Theorem 3.6 follow from their counter-parts in Theorem 3.5, by (3.44). \square

4. EXISTENCE OF \mathcal{B}_{n-2} -SOLUTIONS

We will write $(\zeta_1, \dots, \zeta_k) \in \mathcal{O}$ if given k points $\zeta_1, \dots, \zeta_k \in \mathbb{T}$ are counter clockwise oriented on \mathbb{T} . For example, if $\zeta_1 = 1$, then $(1, \zeta_2, \dots, \zeta_k) \in \mathcal{O}$ means that $\arg \zeta_{i+1} > \arg \zeta_i$ for all $i = 1, \dots, k-1$. From now on, we will assume that the interpolation nodes t_1, \dots, t_n in problem (1.2) are counter clockwise oriented.

Theorem 4.1. *The problem (1.2) has a non-constant solution $f \in \mathcal{B}_{n-2}$ if and only if there exist three target values w_i, w_j, w_k having the same orientation as t_i, t_j, t_k on \mathbb{T} .*

As was pointed out in [21], the “only if” part follows by the winding number argument: the absence of the requested triple means that (up to rotation of \mathbb{T}) $\arg w_n \leq \arg w_{n-1} \leq \dots \leq \arg w_1$ with at least one strict inequality, and then the degree of any Blaschke product interpolation this data is at least $n-1$. In this section we prove the “if” part in Theorem 4.1.

Lemma 4.2. *Given three points $\zeta_i = e^{i\vartheta_i}$ ($i = 1, 2, 3$), the quantity*

$$G(\zeta_1, \zeta_2, \zeta_3) := -i(1 - \zeta_1 \bar{\zeta}_2)(1 - \zeta_2 \bar{\zeta}_3)(1 - \zeta_3 \bar{\zeta}_1)$$

is real. Moreover,

$$G(\zeta_1, \zeta_2, \zeta_3) > 0 \iff (\zeta_1, \zeta_2, \zeta_3) \in \mathcal{O}, \quad G(\zeta_1, \zeta_2, \zeta_3) < 0 \iff (\zeta_1, \zeta_3, \zeta_2) \in \mathcal{O}.$$

Proof. Since G is rotation-invariant, we may assume without loss of generality that $\zeta_1 = 1$ (i.e., $\vartheta_1 = 0$). Then a straightforward computation shows that

$$G(1, \zeta_2, \zeta_3) = 8 \sin \frac{\vartheta_3 - \vartheta_2}{2} \sin \frac{\vartheta_2}{2} \sin \frac{\vartheta_3}{2},$$

which implies all the desired statements. \square

Corollary 4.3. *The product of any three off-diagonal entries p_{ij}, p_{jk}, p_{ki} in the matrix (2.4),*

$$p_{ij} p_{jk} p_{ki} = \frac{1 - w_i \bar{w}_j}{1 - t_i \bar{t}_j} \cdot \frac{1 - w_j \bar{w}_k}{1 - t_j \bar{t}_k} \cdot \frac{1 - w_k \bar{w}_i}{1 - t_k \bar{t}_i} = \frac{G(w_i, w_j, w_k)}{G(t_i, t_j, t_k)}$$

is positive if and only if w_i, w_j, w_k are all distinct and have the same orientation on \mathbb{T} as t_i, t_j, t_k .

Proof of Theorem 4.1. Let us assume that there are three target values having the same orientation on \mathbb{T} as their respective interpolation nodes. By re-enumerating, we may assume without loss of generality that these values are w_{n-2}, w_{n-1}, w_n so that

$$q := \frac{p_{n-1,n-2}p_{n-2,n}}{p_{n-1,n}} = \frac{p_{n-1,n-2}p_{n-2,n}p_{n,n-1}}{|p_{n-1,n}|^2} > 0, \quad (4.1)$$

by Corollary 4.3 and since $p_{n-1,n} = \bar{p}_{n,n-1}$. We will show that in this case, there is an admissible tuple γ so that the number Δ_{n-1}^γ defined in (3.41) equals zero. To this end, let

$$\mathbf{b} = \begin{bmatrix} p_{1,n-2} \\ \vdots \\ p_{n-3,n-2} \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} p_{1,n-1} \\ \vdots \\ p_{n-3,n-1} \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} p_{1,n} \\ \vdots \\ p_{n-3,n} \end{bmatrix}.$$

Let $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\}$ be any admissible tuple. Then the matrix P_{n-1}^γ (3.2) and the column \mathbf{p}_n (3.40) can be written as

$$P_{n-1}^\gamma = \begin{bmatrix} P_{n-3}^\gamma & \mathbf{b} & \mathbf{c} \\ \mathbf{b}^* & \gamma_{n-2} & p_{n-2,n-1} \\ \mathbf{c}^* & p_{n-1,n-2} & \gamma_{n-1} \end{bmatrix} \quad \text{and} \quad \mathbf{p}_n = \begin{bmatrix} \mathbf{d} \\ p_{n-2,n} \\ p_{n-1,n} \end{bmatrix}. \quad (4.2)$$

Replacing the rightmost column in P_{n-1}^γ by \mathbf{p}_n produces

$$P_{n-1,n-1}^\gamma(\mathbf{p}_n) = \begin{bmatrix} P_{n-3}^\gamma & \mathbf{b} & \mathbf{d} \\ \mathbf{b}^* & \gamma_{n-2} & p_{n-2,n} \\ \mathbf{c}^* & p_{n-1,n-2} & p_{n-1,n} \end{bmatrix}. \quad (4.3)$$

For any matrix A , we can make the entries of $(P_{n-3}^\gamma - A)^{-1}$ as small in modulus as we wish by choosing the diagonal entries $\gamma_1, \dots, \gamma_{n-3}$ in P_{n-3}^γ big enough. Thus, we choose $\gamma_1, \dots, \gamma_{n-3}$ so huge that $P_{n-3}^\gamma > 0$,

$$\det(P_{n-3}^\gamma - \mathbf{d}\mathbf{c}^*p_{n-1,n}^{-1}) \neq 0, \quad \mathbf{b}^*(P_{n-3}^\gamma)^{-1}\mathbf{b} < \frac{q}{3}, \quad |X| < \frac{q}{3}, \quad (4.4)$$

where $q > 0$ is specified in (4.1) and where

$$X = \left(\mathbf{b}^* - \frac{p_{n-2,n}}{p_{n-1,n}} \mathbf{c}^* \right) \left(P_{n-3}^\gamma - \mathbf{d}p_{n-1,n}^{-1}\mathbf{c}^* \right)^{-1} \left(\mathbf{b} - \mathbf{d} \frac{p_{n-1,n-2}}{p_{n-1,n}} \right). \quad (4.5)$$

Then we choose

$$\gamma_{n-2} = q + X \quad \text{and} \quad \gamma_{n-1} \geq \mathbf{c}^*(P_{n-3}^\gamma)^{-1}\mathbf{c} + \frac{3}{q} \cdot |p_{n-1,n-2} - \mathbf{c}^*(P_{n-3}^\gamma)^{-1}\mathbf{b}|^2. \quad (4.6)$$

Then the tuple $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\}$ is admissible. Indeed, by (4.4) and (4.6), the Schur complement of P_{n-3}^γ in the matrix $P_{n-2}^\gamma = \begin{bmatrix} P_{n-3}^\gamma & \mathbf{b} \\ \mathbf{b}^* & \gamma_{n-2} \end{bmatrix}$ is positive:

$$\gamma_{n-2} - \mathbf{b}^*(P_{n-3}^\gamma)^{-1}\mathbf{b} = q + X - \mathbf{b}^*(P_{n-3}^\gamma)^{-1}\mathbf{b} > q - \frac{q}{3} - \frac{q}{3} = \frac{q}{3} > 0, \quad (4.7)$$

and therefore, P_{n-2}^γ is positive definite. We next use (4.7) and the second relation in (4.6) to show that the Schur complement of P_{n-2}^γ in the matrix P_{n-1}^γ is also positive:

$$\begin{aligned} & \gamma_{n-1} - [\mathbf{c}^* \quad p_{n-1,n-2}] (P_{n-2}^\gamma)^{-1} \begin{bmatrix} \mathbf{c} \\ p_{n-2,n-1} \end{bmatrix} \\ &= \gamma_{n-1} - \mathbf{c}^* (P_{n-3}^\gamma)^{-1} \mathbf{c} - (\gamma_{n-2} - \mathbf{b}^* (P_{n-3}^\gamma)^{-1} \mathbf{b})^{-1} \cdot |p_{n-1,n-2} - \mathbf{c}^* (P_{n-3}^\gamma)^{-1} \mathbf{b}|^2 \\ &> \gamma_{n-1} - \mathbf{c}^* (P_{n-3}^\gamma)^{-1} \mathbf{c} - \frac{3}{q} \cdot |p_{n-1,n-2} - \mathbf{c}^* (P_{n-3}^\gamma)^{-1} \mathbf{b}|^2 > 0. \end{aligned}$$

Therefore, P_{n-1}^γ is positive definite. Finally, we have from (4.3), (4.1) and (4.5),

$$\begin{aligned} & \det P_{n-1,n-1}^\gamma(\mathbf{p}_n) \\ &= p_{n-1,n} \cdot \det \left(\begin{bmatrix} P_{n-3}^\gamma & \mathbf{b} \\ \mathbf{b}^* & \gamma_{n-2} \end{bmatrix} - \begin{bmatrix} \mathbf{d} \\ p_{n-2,n} \end{bmatrix} [\mathbf{c}^* \quad p_{n-1,n-2}] p_{n-1,n}^{-1} \right) \\ &= p_{n-1,n} \cdot (\gamma_{n-2} - q - X) \cdot \det(P_{n-3}^\gamma - \mathbf{d} \mathbf{c}^* p_{n-1,n}^{-1}) = 0. \end{aligned}$$

where the last equality holds by the choice (4.6) of γ_{n-2} . By formula (3.42), $\Delta_{n-1}^\gamma = 0$. Then formula (3.42) will produce $f_\gamma \in \mathcal{B}_{n-2}$ solving the problem (1.2). This solution is not a constant function since w_{n-2}, w_{n-1}, w_n are all distinct. \square

5. EXAMPLES

In this section we illustrate Theorem 3.5 by several particular examples where the parametrization formula (3.15) (or (3.42)) is particularly explicit in terms of the interpolation data set.

5.1. Three-points problem. (cf. Example 3 in [21]). We want to find all $f \in \mathcal{B}_2$ satisfying conditions

$$f(t_i) = w_i \quad (t_i, f_i \in \mathbb{T}, i = 1, 2, 3). \quad (5.1)$$

We exclude the trivial case where $f_1 = f_2 = f_3$. By Theorem 3.6, all solutions $f \in \mathcal{B}_2$ to the problem (5.1) are given by the formula

$$f^\gamma(z) = w_3 \cdot \frac{1 - (1 - z\bar{t}_3) \cdot \left(\frac{\Delta_1^\gamma}{1 - z\bar{t}_1} + \frac{\Delta_2^\gamma}{1 - z\bar{t}_2} \right)}{1 - (1 - z\bar{t}_3) \cdot \left(\frac{w_3\bar{w}_1\Delta_1^\gamma}{1 - z\bar{t}_1} + \frac{w_3\bar{w}_2\Delta_2^\gamma}{1 - z\bar{t}_2} \right)}, \quad (5.2)$$

where Δ_1^γ and Δ_2^γ are given, according to (3.41), by

$$\Delta_1^\gamma = \frac{\gamma_2 p_{13} - p_{12} p_{23}}{\det P_2^\gamma} \quad \text{and} \quad \Delta_2^\gamma = \frac{\gamma_1 p_{23} - p_{21} p_{13}}{\det P_2^\gamma}, \quad (5.3)$$

where $p_{ij} = \frac{1 - w_i \bar{w}_j}{1 - t_i \bar{t}_j}$ and where $\gamma = \{\gamma_1, \gamma_2\}$ is any admissible pair, i.e.,

$$\gamma_1 > 0, \quad \gamma_2 > 0, \quad \gamma_1 \gamma_2 > |p_{12}|^2.$$

Thus, any point (γ_1, γ_2) in the first quadrant \mathbb{R}_+^2 above the graph of $y = |p_{12}|^2 x^{-1}$ gives rise via formula (5.2) to a \mathcal{B}_2 -solution to the problem (5.1). It follows immediately by the winding number argument that there are no solutions of

degree one if w_1, w_2, w_3 do not have the same orientation on \mathbb{T} as t_1, t_2, t_3 . We can come to the same conclusion showing that in this case,

$$\Delta_1^\gamma \neq 0 \quad \text{and} \quad \Delta_2^\gamma \neq 0 \quad (5.4)$$

for any admissible $\{\gamma_1, \gamma_2\}$. Indeed, if $p_{13} = 0$ (i.e., $w_1 = w_3$), then (5.4) holds since p_{12}, p_{23} and γ_1 are all non-zero. Similarly, (5.4) holds if $p_{23} = 0$. If $p_{13} \neq 0$ and $p_{23} \neq 0$, then the numbers

$$\tilde{\gamma}_2 := \frac{p_{12}p_{23}}{p_{13}} = \frac{p_{12}p_{23}p_{31}}{|p_{13}|^2} \quad \text{and} \quad \tilde{\gamma}_1 := \frac{p_{21}p_{13}}{p_{23}} = \frac{p_{21}p_{13}p_{32}}{|p_{23}|^2} \quad (5.5)$$

are both non-positive, by Corollary 4.3. Hence, inequalities (5.4) hold for any positive γ_1, γ_2 and therefore, there are no zero cancellations in (5.2). On the other hand, if w_1, w_2, w_3 have the same orientation on \mathbb{T} as t_1, t_2, t_3 , then the numbers $\tilde{\gamma}_2$ and $\tilde{\gamma}_1$ in (5.5) are positive (again, by Corollary 4.3). Observe that $\tilde{\gamma}_1\tilde{\gamma}_2 = |p_{12}|^2$. Therefore, any pair $(\gamma_1, \tilde{\gamma}_2)$ with $\gamma_1 > \tilde{\gamma}_1$ is admissible, and since

$$\Delta_1^{\{\gamma_1, \tilde{\gamma}_2\}} = 0, \quad \Delta_2^{\{\gamma_1, \tilde{\gamma}_2\}} = \frac{\gamma_1 p_{23} - p_{21} p_{13}}{\gamma_1 \frac{p_{12} p_{23}}{p_{13}} - |p_{12}|^2} = \frac{p_{13}}{p_{12}},$$

the formula (5.2) amounts to

$$f^{\{\gamma_1, \tilde{\gamma}_2\}}(z) = \frac{(1 - z\bar{t}_2)p_{12} - (1 - z\bar{t}_3)p_{13}}{(1 - z\bar{t}_2)\bar{w}_3 p_{12} - (1 - z\bar{t}_3)\bar{w}_2 p_{13}} \quad (5.6)$$

for any $\gamma_1 > \tilde{\gamma}_1$. On the other hand, any pair $(\tilde{\gamma}_1, \gamma_2)$ with $\gamma_2 > \tilde{\gamma}_2$ is admissible, and since now

$$\Delta_1^{\{\tilde{\gamma}_1, \gamma_2\}} = \frac{\gamma_2 p_{13} - p_{12} p_{23}}{\gamma_2 \frac{p_{21} p_{13}}{p_{23}} - |p_{12}|^2} = \frac{p_{23}}{p_{21}} \quad \text{and} \quad \Delta_2^{\{\tilde{\gamma}_1, \gamma_2\}} = 0,$$

the formula (5.2) amounts to

$$f^{\{\tilde{\gamma}_1, \gamma_2\}}(z) = \frac{(1 - z\bar{t}_1)p_{21} - (1 - z\bar{t}_3)p_{23}}{(1 - z\bar{t}_1)\bar{w}_3 p_{21} - (1 - z\bar{t}_3)\bar{w}_1 p_{23}} \quad (5.7)$$

for any $\gamma_2 > \tilde{\gamma}_2$. A straightforward verification shows that formulas (5.6) and (5.7) define the same function (as expected, since the problem (5.1) has at most one rational solution of degree one).

5.2. Another example. We next consider the n -point problem (1.2) where all target values but one are equal to each other (we assume without loss of generality that this common value is 1):

$$f(t_i) = 1 \quad (i = 1, \dots, n-1) \quad \text{and} \quad f(t_n) = w_n \in \mathbb{T} \setminus \{1\}. \quad (5.8)$$

Proposition 5.1. *All functions $f \in \mathcal{B}_{n-1}^\circ$ subject to interpolation conditions (5.8) are parametrized by the formula*

$$f(z) = w_n \cdot \frac{1 - \sum_{i=1}^{n-1} \frac{(1 - z\bar{t}_n)(1 - \bar{w}_n)}{(1 - z\bar{t}_i)(1 - t_i\bar{t}_n)\gamma_i}}{1 - \sum_{i=1}^{n-1} \frac{(1 - z\bar{t}_n)(w_n - 1)\bar{w}_i}{(1 - z\bar{t}_i)(1 - t_i\bar{t}_n)\gamma_i}}, \quad (5.9)$$

where positive numbers $\gamma_1, \dots, \gamma_{n-1}$ are free parameters.

Proof. For the data set as in (5.8), $p_{i,j} = 0$ for all $1 \leq i \neq j \leq n-1$. Therefore, the matrix P_{n-1}^γ is diagonal and $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\}$ is admissible if and only if $\gamma_i > 0$ ($1 \leq i \leq n-1$). Furthermore, the numbers (3.41) are equal to

$$\Delta_i^\gamma = \frac{p_{i,n}}{\gamma_i} = \frac{1 - w_i \bar{w}_n}{\gamma_i(1 - t_i \bar{t}_n)} = \frac{1 - \bar{w}_n}{\gamma_i(1 - t_i \bar{t}_n)} \quad \text{for } i = 1, \dots, n-1.$$

Substituting the latter formulas in (3.42) gives (5.9). By Theorem 3.5, formula (5.9) parametrizes all \mathcal{B}_{n-1} -solutions to the problem (5.8). However, since $\Delta_i^\gamma \neq 0$ for all $i = 1, \dots, n-1$, it follows that $\deg f = n-1$ for any f of the form (5.9). \square

Remark 5.2. Letting $w_n = 1$ in (5.9) we see that the only $f \in \mathcal{B}_{n-1}$ subject to equalities $f(t_i) = 1$ for $i = 1, \dots, n$, is the constant function $f(z) \equiv 1$.

We now show how to get formula (5.9) using the approach from [5, 9] as follows. Let $f \in \mathcal{B}_{n-1}^\circ$ satisfy conditions (5.8) and let $\gamma_i = |f'(t_i)|$ ($1 \leq i \leq n-1$). Then $f^{-1}(\{1\}) = \{t_1, \dots, t_{n-1}\}$ and the Aleksandrov-Clark measure $\mu_{f,1}$ of f at 1 is the sum of n point masses γ_i^{-1} at t_i . Therefore,

$$\frac{1 + f(z)}{1 - f(z)} = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu_{f,1}(\zeta) + ic = \sum_{i=1}^{n-1} \frac{1}{\gamma_i} \cdot \frac{t_i + z}{t_i - z} + ic \quad (5.10)$$

for some $c \in \mathbb{R}$. Solving (5.10) for f and letting $\mathcal{E} = \frac{ic-1}{ic+1}$ (note that $\mathcal{E} \in \mathbb{T} \setminus \{1\}$) we get

$$f(z) = \frac{(1 - \Phi(z))\mathcal{E} + \Phi(z)}{-\Phi(z)\mathcal{E} + 1 + \Phi(z)}, \quad \text{where } \Phi(z) = \frac{1}{2} \cdot \sum_{i=1}^{n-1} \frac{1}{\gamma_i} \cdot \frac{t_i + z}{t_i - z}. \quad (5.11)$$

Evaluating (5.11) at $z = t_n$ and making use of the last condition in (5.8) we get

$$w_n = \frac{(1 - \Phi(t_n))\mathcal{E} + \Phi(t_n)}{-\Phi(t_n)\mathcal{E} + 1 + \Phi(t_n)} \iff \mathcal{E} = \frac{(1 + \Phi(t_n))w_n - \Phi(t_n)}{\Phi(t_n)w_n + 1 - \Phi(t_n)}.$$

Substituting the latter expression for \mathcal{E} into (5.11) leads us to the representation

$$f(z) = \frac{w_n + (\Phi(z) - \Phi(t_n))(1 - w_n)}{1 + (\Phi(z) - \Phi(t_n))(1 - w_n)}$$

which is the same as (5.9), since $|w_n| = 1$ and since according to (5.11),

$$\Phi(z) - \Phi(t_n) = \frac{1}{2} \cdot \sum_{i=1}^{n-1} \frac{1}{\gamma_i} \cdot \left(\frac{t_i + z}{t_i - z} - \frac{t_i + t_n}{t_i - t_n} \right) = \sum_{i=1}^{n-1} \frac{1 - z \bar{t}_n}{\gamma_i (1 - z \bar{t}_i)(1 - t_i \bar{t}_n)}.$$

5.3. Boundary fixed points. By Theorem 1.1, there are infinitely many finite Blaschke products $f \in \mathcal{B}_{n-1}$ with given fixed boundary points $t_1, \dots, t_n \in \mathbb{T}$, i.e., such that

$$f(t_i) = t_i \quad \text{for } i = 1, \dots, n. \quad (5.12)$$

We will use Theorem 3.5 to parametrize all such Blaschke products. Since $w_i = t_i$ for $i = 1, \dots, n$, we have $p_{i,j} = 1$ for all $1 \leq i \neq j \leq n$. By definition (3.2),

$$P_{n-1}^\gamma = \Gamma + EE^*, \quad \text{where} \quad \Gamma = \begin{bmatrix} \gamma_1 - 1 & & 0 \\ & \ddots & \\ 0 & & \gamma_{n-1} - 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \quad (5.13)$$

If the tuple $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\}$ contains two elements $\gamma_i \leq 1$ and $\gamma_j \leq 1$, then it is not admissible since the principal submatrix $\begin{bmatrix} \gamma_1 & 1 \\ 1 & \gamma_2 \end{bmatrix}$ of P_{n-1}^γ is not positive definite. Otherwise, that is, in one of the three following cases, the tuple γ is admissible:

- (1) $\gamma_i > 1$ for all $i \in \{1, \dots, n-1\}$.
- (2) $\gamma_\ell = 1$ and $\gamma_i > 1$ for all $i \neq \ell$.
- (3) $\gamma_\ell < 1$, $\gamma_i > 1$ for all $i \neq \ell$, and $\det P_{n-1}^\gamma > 0$.

Cases 1&3: Since $\gamma_i \neq 0$ for $i = 1, \dots, n$, the matrix Γ is invertible. By basic properties of determinants,

$$\det P_{n-1}^\gamma = \det \Gamma \cdot \det(I + \Gamma^{-1}EE^*) = \det \Gamma \cdot (1 + E^*\Gamma^{-1}E),$$

which, on account (5.13), implies

$$\det P_{n-1}^\gamma = \left(1 + \sum_{i=1}^{n-1} \frac{1}{\gamma_i - 1}\right) \cdot \prod_{i=1}^{n-1} (\gamma_i - 1). \quad (5.14)$$

Using the latter formula, we can characterize Case 3 as follows:

$$\gamma_\ell < 1, \quad \gamma_i > 1 \quad \text{for all} \quad i \neq \ell, \quad \text{and} \quad \sum_{i=1}^{n-1} \frac{1}{\gamma_i - 1} < -1. \quad (5.15)$$

Since $p_{i,n} = 1$ for all $1 \leq i \leq n-1$, we have $\mathbf{p}_n = E$ in (3.40), and hence,

$$\det P_{n-1,j}^\gamma(E) = \lim_{\gamma_j \rightarrow 1} \det P_{n-1}^\gamma = \prod_{i \neq j} (\gamma_i - 1) \quad \text{for} \quad j = 1, \dots, n-1.$$

Substituting the two latter formulas in (3.41) we get

$$\Delta_j^\gamma = \frac{\det P_{n-1,j}^\gamma(E)}{\det P_{n-1}^\gamma} = \frac{1}{\left(1 + \sum_{i=1}^{n-1} \frac{1}{\gamma_i - 1}\right) \cdot (\gamma_j - 1)} \quad (j = 1, \dots, n-1). \quad (5.16)$$

Substituting (5.16) into (3.42) (with $w_i = t_i$ for $i = 1, \dots, n-1$) leads us (after straightforward algebraic manipulations) to the formula

$$f(z) = \frac{t_n + \sum_{i=1}^{n-1} \frac{z(1 - t_n \bar{t}_i)}{(1 - z \bar{t}_i)(\gamma_i - 1)}}{1 + \sum_{i=1}^{n-1} \frac{1 - t_n \bar{t}_i}{(1 - z \bar{t}_i)(\gamma_i - 1)}}. \quad (5.17)$$

Since $\Delta_j^\gamma \neq 0$ for all $j = 1, \dots, n-1$, it follows by Theorem 3.5 that f is a Blaschke product of degree $n-1$ and $|f'(t_i)| = f'(t_i) = \gamma_i$ for $i = 1, \dots, n-1$. Differentiating (5.17) and evaluating the obtained formula for $f'(z)$ at $z = t_n$ gives

$$\gamma_n := f'(t_n) = \left(\sum_{i=1}^{n-1} \frac{1}{\gamma_i - 1} \right) \left(1 + \sum_{i=1}^{n-1} \frac{1}{\gamma_i - 1} \right)^{-1},$$

which can be equivalently written as

$$\frac{\gamma_n}{1 - \gamma_n} = \sum_{i=1}^{n-1} \frac{1}{\gamma_i - 1} \quad \text{or} \quad \sum_{i=1}^n \frac{1}{\gamma_i - 1} = -1. \quad (5.18)$$

It follows from (5.18) that in Case 1, $0 < \gamma_n < 1$, so that t_n is the (hyperbolic) Denjoy-Wolff point of f . Alternatively, this conclusion follows from the Cowen-Pommerenke result [18]: *the inequality*

$$\sum_{i=1}^{n-1} \frac{1}{f'(t_i) - 1} \leq \frac{f'(t_n)}{1 - f'(t_n)} \quad (5.19)$$

for any analytic $f : \mathbb{D} \rightarrow \mathbb{D}$ with boundary fixed points t_1, \dots, t_{n-1} and the (hyperbolic) boundary fixed point t_n , and equality prevails in (5.19) if and only if $f \in \mathcal{B}_{n-1}^\circ$. Observe, that in Case 3, the point t_ℓ is the Denjoy-Wolff point of f , while t_n is a regular boundary fixed point with $\gamma_n = f'(t_n) > 1$.

Case 2: Direct computations show that in this case,

$$\det P_{n-1}^\gamma = \prod_{j \neq \ell} (\gamma_j - 1) = \det P_{n-1, \ell}^\gamma(E) \quad \text{and} \quad \det P_{n-1, i}^\gamma(E) = 0 \quad (i \neq \ell).$$

Therefore, according to (3.41), $\Delta_\ell^\gamma = 1$ and $\Delta_i^\gamma = 0$ for all $i \neq \ell$, which being substituted into (3.42), gives

$$f(z) = t_n \cdot \frac{1 - \frac{1-z\bar{t}_n}{1-z\bar{t}_\ell}}{1 - \frac{1-z\bar{t}_n}{1-z\bar{t}_\ell} \bar{t}_\ell t_n} = z.$$

We summarize the preceding analysis in the next proposition.

Proposition 5.3. *All functions $f \in \mathcal{B}_{n-1}^\circ$ satisfying conditions (5.8) are given by the formula (5.17), where the parameter $\gamma = \{\gamma_1, \dots, \gamma_{n-1}\}$ is either subject to relations (5.15) (in which case t_ℓ is the Denjoy-Wolff point of f) or $\gamma_i > 1$ for all i (in which case t_n is the Denjoy-Wolff point of f).*

In particular, it follows that the identity mapping $f(z) = z$ is the only function in \mathcal{B}_{n-2} satisfying conditions (5.12). In fact, a more general result in [1] asserts that if the n -point Nevanlinna-Pick problem (1.2) (with $t_i, w_i \in \mathbb{C}$) has a rational solution of degree $k < n/2$ (in the setting of (5.12), $k = 1$), then it does not have other rational solutions of degree less than $n - k$.

6. CONCLUDING REMARKS AND OPEN QUESTIONS

Some partial results on the problem (1.2) can be derived from general results on rational interpolation. Let $[x]$ denote the integer part of $x \in \mathbb{R}$ (the greatest integer not exceeding a given x). If we let

$$q = \text{rank} \left[\frac{w_{r+j} - w_i}{t_{r+i} - t_j} \right]_{i,j=1}^r = \text{rank} \left[\frac{1 - w_i \bar{w}_{r+j}}{1 - t_i \bar{t}_{r+j}} \right]_{i,j=1}^r, \quad r = \left[\frac{n}{2} \right], \quad (6.1)$$

then, by a result from [1], there are no rational functions f of degree less than q satisfying conditions (1.2). If $q \leq \frac{n-1}{2}$, then there is at most one rational f of degree equal q satisfying conditions (1.2), and the only candidate can be found in the form

$$f(z) = \frac{a_0 + a_1 z + \dots + a_q z^q}{b_0 + b_1 z + \dots + b_q z^q} \quad (6.2)$$

by solving the linear system

$$a_0 + a_1 t_i + \dots + a_q t_i^q = w_i (b_0 + b_1 t_i + \dots + b_q t_i^q), \quad i = 1, \dots, n.$$

If there are no zero cancellations in the representation (6.2), f satisfies all conditions in (1.2). This f is a finite Blaschke product if and only if it has no poles in \bar{D} and $b_i = \bar{a}_{q-i}$ for $i = 1, \dots, n$. Alternatively, $f \in \mathcal{B}_q^\circ$ if and only if the boundary Schwarz-Pick matrix $P^f(t_1, \dots, t_n)$ (see (2.1)) is positive semidefinite. Another result from [1] states that the next possible degree of a rational solution to the problem (1.2) is $n - q$ and moreover, there are infinitely many rational solutions of degree k for each $k \geq n - q$. Simple examples show that finite Blaschke product solutions of degree $n - q$ may not exist, so that the result from [1] provides a *lower bound* for the minimally possible degree of a Blaschke product solution. On the other hand (see e.g., [21]), if the problem (1.2) has a solution in $\mathcal{B}_{\kappa_0}^\circ$ for $\kappa_0 > \frac{n-1}{2}$, then it has infinitely solutions in \mathcal{B}_k° for each $k \geq \kappa_0$. Hence, the procedure verifying whether or not the problem (1.2) has a unique minimal degree Blaschke product solution (necessarily, $\deg f \leq \frac{n-1}{2}$) is simple. The question of some interest is to characterize the latter determinate case in terms of the original interpolation data set. A much more interesting question is:

Question 1: Find the minimally possible $\kappa_0 > \frac{n-1}{2}$ so that the problem (1.2) has a solution in $\mathcal{B}_{\kappa_0}^\circ$. For each $k \geq \kappa$, parametrize all \mathcal{B}_k° -solutions to the problem.

In [12, 13], the boundary problem (1.2) was considered in the set \mathcal{QB}_{n-1} of rational functions of degree at most $n - 1$ that are unimodular on \mathbb{T} . Observe that any element of \mathcal{QB}_{n-1} is equal to the ratio of two coprime Blaschke products g, h such that $\deg g + \deg h \leq n - 1$. In general, it is not true that a rational function f of degree less than n and taking unimodular values at n points on \mathbb{T} is necessarily unimodular on \mathbb{T} (by Theorem 1.1, this is true, if a priori, f is subject to $|f(t)| \leq 1$ for all $t \in \mathbb{T}$). Hence, the results concerning low-degree unimodular interpolation do not follow directly from the general results on the unconstrained rational interpolation. However, we were not able to find an example providing the negative answer for the next question:

Question 2: Let q be defined as in (6.1), so that there are infinitely many rational functions f , $\deg f = n - q$, satisfying conditions (1.2). Is it true that some (and therefore, infinitely many) of them are unimodular on \mathbb{T} ?

We finally reformulate the boundary Nevanlinna-Pick problem (1.2) in terms of a positive semidefinite matrix completion problem. With t_1, \dots, t_n and w_1, \dots, w_n in hands, we specify all off-diagonal entries in the matrix P_n :

$$P_n = \begin{bmatrix} * & \frac{1-w_1\bar{w}_2}{1-t_1\bar{t}_2} & \cdots & \frac{1-w_1\bar{w}_{n-1}}{1-t_1\bar{t}_{n-1}} & \frac{1-w_1\bar{w}_n}{1-t_1\bar{t}_n} \\ \frac{1-w_2\bar{w}_1}{1-t_2\bar{t}_1} & * & \cdots & \frac{1-w_2\bar{w}_{n-1}}{1-t_2\bar{t}_{n-1}} & \frac{1-w_2\bar{w}_n}{1-t_2\bar{t}_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1-w_{n-1}\bar{w}_1}{1-t_{n-1}\bar{t}_1} & \frac{1-w_{n-1}\bar{w}_2}{1-t_{n-1}\bar{t}_2} & \cdots & * & \frac{1-w_{n-1}\bar{w}_n}{1-t_{n-1}\bar{t}_n} \\ \frac{1-w_n\bar{w}_1}{1-t_n\bar{t}_1} & \frac{1-w_n\bar{w}_2}{1-t_n\bar{t}_2} & \cdots & \frac{1-w_n\bar{w}_{n-1}}{1-t_n\bar{t}_{n-1}} & * \end{bmatrix}. \quad (6.3)$$

Every choice of the (ordered) set $\gamma = \{\gamma_1, \dots, \gamma_n\}$ of real diagonal entries produces a Hermitian completion of P_n which we have denoted by P_n^γ in (3.2). Completion question related to the problem (1.2) and to a similar problem in the class \mathcal{QB}_{n-1} are the following:

Question 3: Given a partially specified matrix P_n (6.3), find a positive semi-definite completion P_n^γ with minimal possible rank.

The same question concerning finding the minimal rank *Hermitian* completion P_n^γ is related to minimal degree boundary interpolation by unimodular functions. Once the minimal rank completion is found, the finite Blaschke product f with the boundary Schwarz-Pick matrix $P^f(\mathbf{t}) = P_n^\gamma$ will be the minimal degree solution to the problem (1.2). Although Question 3 looks like an exercise on linear algebra, it turns out as difficult as the original interpolation problem.

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